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Some twisted results

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Abstract

The Drinfeld twist for the opposite quasi-Hopf algebra, H^{cop}, is determined and is shown to be related to the (second) Drinfeld twist on a quasi-Hopf algebra. The twisted form of the Drinfeld twist is investigated. In the quasi-triangular case, it is shown that the Drinfeld u-operator arises from the equivalence of H^{cop} to the quasi-Hopf algebra induced by twisting H with the R-matrix. The Altschuler–Coste u-operator arises in a similar way and is shown to be closely related to the Drinfeld u-operator. The quasi-cocycle condition is introduced and is shown to play a central role in the uniqueness of twisted structures on quasi-Hopf algebras. A generalization of the dynamical quantum Yang–Baxter equation, called the quasi-dynamical quantum Yang–Baxter equation, is introduced.

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1. Introduction

Quasi-Hopf algebras (QHA) were introduced by Drinfeld [6] as generalizations of Hopf algebras. QHA are the underlying algebraic structures of elliptic quantum groups [8–11, 14, 20] and hence have an important role in obtaining solutions to the dynamical Yang–Baxter equation. They arise in conformal field theory [3, 4], algebraic number theory [7] and in the theory of knots [1, 15, 16].

The antipode S of a Hopf algebra H is uniquely determined as the inverse of the identity map on H under the convolution product. For a quasi-Hopf algebra, the triple (S, α, β) consisting of the antipode S and canonical elements $\alpha, \beta \in H$ is termed the *quasi-antipode*. The quasi-antipode of a QHA is not unique [2, 6, 17]. However, given two QHAs which differ only in their quasi-antipodes, there exists a unique invertible element $v \in H$ relating them. Moreover, to each invertible element $v \in H$ there corresponds a quasi-antipode, so that the invertible elements $v \in H$ are in bijection with the quasi-antipodes. This allows us to work with a fixed choice for the quasi-antipode (more precisely, a fixed equivalence class for the quasi-antipode). We show that the operator $v \in H$ is universal, i.e. invariant under an arbitrary twist $F \in H \otimes H$. In the quasi-triangular case, the equivalence of the quasi-antipode of the opposite QHA H^{cop} and the quasi-antipode induced by twisting H with the R-matrix gives rise to a specific form of the v operator, which we call the Drinfeld–Reshetikhin [5, 18] u-operator. The u-operator introduced by Altschuler and Coste [1] in the context of ribbon quasi-Hopf algebras arises in a similar way and is shown to be simply related to the Drinfeld–Reshetikhin u-operator. In view of the invariance of the v operators, these u-operators are also invariant under twisting.

For a Hopf algebra H, the antipode S is both an algebra and a co-algebra anti-homomorphism. In the QHA case, Drinfeld has shown that the antipode S is a co-algebra anti-homomorphism only upto conjugation by a twist, F_δ (the Drinfeld twist). Assuming the antipode S is invertible with inverse S^{-1}, we show that S^{-1} is a co-algebra anti-homomorphism upto conjugation by an invertible element F_0, which we call the second Drinfeld twist on H. The form of the Drinfeld twist for the opposite QHA H^{cop} is determined and shown to be simply related to this second Drinfeld twist. The behaviour of the Drinfeld twist F_δ under an arbitrary twist $G \in H \otimes H$ is also investigated.

The set of twists on a QHA H form a group. We study a subgroup of the group of twists on a QHA, namely those that leave the co-product $\Delta : H \to H \otimes H$ and the co-associator $\Phi \in H \otimes H \otimes H$ unchanged. These twists are called *compatible twists*. Twists that leave the co-associator Φ unchanged are said to satisfy the quasi-cocycle condition. The quasi-cocycle condition is intimately related to the uniqueness of the structure obtained by twisting the quasi-bialgebra part of a QHA. In the quasi-triangular case, we show that $\mathcal{R}^T\mathcal{R}$ and its powers are compatible twists.

Following on from our considerations of the quasi-cocycle condition, we introduce the shifted quasi-cocycle condition on a twist $F(\lambda) \in H \otimes H$, where $\lambda \in H$ depends on one (or more) parameter(s). We conclude with the quasi-dynamical quantum Yang–Baxter equation (QQYBE), which is the quasi-Hopf analogue of the usual dynamical QYBE.

2. Preliminaries

We begin by recalling the definition [6] of a quasi-bialgebra.

Definition 1. *A quasi-bialgebra* $(H, \Delta, \epsilon, \Phi)$ *is a unital associative algebra* H *over a field* F*, equipped with algebra homomorphisms* $\epsilon : H \to F$ *(co-unit),* $\Delta : H \to H \otimes H$ *(co-product) and an invertible element* $\Phi \in H \otimes H \otimes H$ *(co-associator) satisfying*

$$(1 \otimes \Delta)\Delta(a) = \Phi^{-1}(\Delta \otimes 1)\Delta(a)\Phi, \qquad \forall a \in H, \tag{2.1}$$

$$(\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi = (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)\Phi \cdot (1 \otimes \Phi), \tag{2.2}$$

$$(\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta, \tag{2.3}$$

$$(1 \otimes \epsilon \otimes 1)\Phi = 1. \tag{2.4}$$

It follows from equations (2.2)–(2.4) that the co-associator Φ has the additional properties

$$(\epsilon \otimes 1 \otimes 1)\Phi = 1 = (1 \otimes 1 \otimes \epsilon)\Phi.$$

We now fix the notation to be used throughout the paper. For the co-associator, we follow the notation of [12, 13] and write

$$\Phi = \sum_\nu X_\nu \otimes Y_\nu \otimes Z_\nu, \qquad \Phi^{-1} = \sum_\nu \bar{X}_\nu \otimes \bar{Y}_\nu \otimes \bar{Z}_\nu.$$

We adopt Sweedler's [19] notation for the co-product

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}, \qquad \forall a \in H$$

throughout. Since the co-product is quasi-co-associative, we use the following extension of Sweedler's notation:

$$\begin{aligned} (1 \otimes \Delta)\Delta(a) &= a_{(1)} \otimes \Delta(a_{(2)}) = a_{(1)} \otimes a_{(2)}^{(1)} \otimes a_{(2)}^{(2)}, \\ (\Delta \otimes 1)\Delta(a) &= \Delta(a_{(1)}) \otimes a_{(2)} = a_{(1)}^{(1)} \otimes a_{(1)}^{(2)} \otimes a_{(2)}. \end{aligned} \tag{2.5}$$

In general, the summation sign is omitted from expressions with the convention that repeated indices are to be summed over.

Definition 2. *A quasi-Hopf algebra* $(H, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ *is a quasi-bialgebra* $(H, \Delta, \epsilon, \Phi)$ *equipped with an algebra anti-homomorphism S (antipode) and canonical elements* $\alpha, \beta \in H$*, such that*

$$S(X_\nu)\alpha Y_\nu \beta S(Z_\nu) = 1 = \bar{X}_\nu \beta S(\bar{Y}_\nu)\alpha \bar{Z}_\nu, \tag{2.6}$$

$$S(a_{(1)})\alpha a_{(2)} = \epsilon(a)\alpha, \qquad a_{(1)}\beta S(a_{(2)}) = \epsilon(a)\beta, \qquad \forall a \in H. \tag{2.7}$$

Throughout we assume bijectivity of the antipode S so that S^{-1} exists. The antipode equations (2.6), (2.7) imply $\epsilon(\alpha) \cdot \epsilon(\beta) = 1$ and $\epsilon(S(a)) = \epsilon(S^{-1}(a)) = \epsilon(a), \forall a \in H$. A triple (S, α, β) satisfying equations (2.6), (2.7) is called a *quasi-antipode*.

We shall need the following relations:

$$X_\nu a \otimes Y_\nu \beta S(Z_\nu) = a_{(1)}^{(1)} X_\nu \otimes a_{(1)}^{(2)} Y_\nu \beta S(Z_\nu) S(a_{(2)}), \qquad \forall a \in H, \tag{2.8}$$

$$\begin{aligned} \Phi \otimes 1 &\stackrel{(2.2)}{=} (\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi \cdot (1 \otimes \Phi^{-1}) \cdot (1 \otimes \Delta \otimes 1)\Phi^{-1} \\ &= X_\nu^{(1)} X_\mu \bar{X}_\rho \otimes X_\nu^{(2)} Y_\mu \bar{X}_\sigma \bar{Y}_\rho^{(1)} \otimes Y_\nu Z_\mu^{(1)} \bar{Y}_\sigma \bar{Y}_\rho^{(2)} \otimes Z_\nu Z_\mu^{(2)} \bar{Z}_\sigma \bar{Z}_\rho, \end{aligned} \tag{2.9}$$

$$\begin{aligned} 1 \otimes \Phi &= (1 \otimes \Delta \otimes 1)\Phi^{-1} \cdot (\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi, \\ &= \bar{X}_\nu \bar{X}_\mu X_\rho^{(1)} X_\sigma \otimes \bar{Y}_\nu^{(1)} \bar{Y}_\mu X_\rho^{(2)} Y_\sigma \otimes \bar{Y}_\nu^{(2)} \bar{Z}_\mu Y_\rho Z_\sigma^{(1)} \otimes \bar{Z}_\nu Z_\rho Z_\sigma^{(2)}, \end{aligned} \tag{2.10}$$

where we have adopted the notation of equation (2.5) into (2.8) and the obvious notation in (2.9), (2.10) so that, for example

$$\Delta(X_\nu) = X_\nu^{(1)} \otimes X_\nu^{(2)}, \quad \text{etc.}$$

Equation (2.8) follows from applying $(1 \otimes m)(1 \otimes 1 \otimes \beta S)$ to equation (2.1) then using (2.7).

3. Uniqueness of the quasi-antipode

For Hopf algebras, the antipode S is uniquely determined as the inverse of the identity map on H under the convolution product. The quasi-antipode (S, α, β) for a QHA is not unique. Nevertheless, it is almost unique as the following result due to Drinfeld [6] (whose proof is similar to the one given below) shows:

Theorem 1. *Suppose H is also a QHA, but with quasi-antipode* $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$ *satisfying (2.6), (2.7). Then there exists a unique invertible* $v \in H$*, such that*

$$v\alpha = \tilde{\alpha}, \qquad \tilde{\beta} v = \beta, \qquad \tilde{S}(a) = vS(a)v^{-1}, \qquad \forall a \in H. \tag{3.1}$$

Explicitly

$$\begin{aligned} &\text{(i)} \quad v = \tilde{S}(X_\nu)\tilde{\alpha} Y_\nu \beta S(Z_\nu) = \tilde{S}(S^{-1}(\bar{X}_\nu))\tilde{S}(S^{-1}(\beta))\tilde{S}(\bar{Y}_\nu)\tilde{\alpha}\bar{Z}_\nu, \\ &\text{(ii)} \quad v^{-1} = S(X_\nu)\alpha Y_\nu \tilde{\beta}\tilde{S}(Z_\nu) = \bar{X}_\nu \tilde{\beta}\tilde{S}(\bar{Y}_\nu)\tilde{S}(S^{-1}(\alpha))\tilde{S}(S^{-1}(\bar{Z}_\nu)). \end{aligned} \tag{3.2}$$

Proof. We proceed stepwise.

Applying $m\cdot(\tilde{S}\otimes 1)(1\otimes\tilde{\alpha})$ to equation (2.8) gives

$$\tilde{S}(X_\nu a)\tilde{\alpha}Y_\nu\beta S(Z_\nu)=\tilde{S}\big(a^{(1)}_{(1)}X_\nu\big)\tilde{\alpha}a^{(2)}_{(1)}Y_\nu\beta S(Z_\nu)S(a_{(2)}),$$

so that

$$\tilde{S}(a)v=\tilde{S}(X_\nu)\tilde{S}\big(a^{(1)}_{(1)}\big)\tilde{\alpha}a^{(2)}_{(1)}Y_\nu\beta S(Z_\nu)S(a_{(2)})\overset{(2.7)}{=}vS(a),\qquad\forall a\in H,\tag{3.3}$$

where $m:H\otimes H\to H$ is the multiplication map $m(a\otimes b)=ab,\forall a,b\in H$.

Next observe, from equation (2.9) that, in view of (2.7),

$$\begin{aligned}v\otimes 1&=\tilde{S}\big(X^{(1)}_\nu X_\mu\bar{X}_\rho\big)\tilde{\alpha}X^{(2)}_\nu Y_\mu\bar{X}_\sigma\bar{Y}^{(1)}_\rho\beta S\big(Y_\nu Z^{(1)}_\mu\bar{Y}_\sigma\bar{Y}^{(2)}_\rho\big)\otimes Z_\nu Z^{(2)}_\mu\bar{Z}_\sigma\bar{Z}_\rho\\&=\tilde{S}(X_\mu)\tilde{\alpha}Y_\mu\bar{X}_\sigma\beta S\big(Z^{(1)}_\mu\bar{Y}_\sigma\big)\otimes Z^{(2)}_\mu\bar{Z}_\sigma.\end{aligned}$$

Applying $m\cdot(1\otimes\alpha)$ from the left gives

$$\begin{aligned}v\alpha&=\tilde{S}(X_\mu)\tilde{\alpha}Y_\mu\bar{X}_\sigma\beta S\big(Z^{(1)}_\mu\bar{Y}_\sigma\big)\alpha Z^{(2)}_\mu\bar{Z}_\sigma\\&=\tilde{\alpha}\bar{X}_\sigma\beta S(\bar{Y}_\sigma)\alpha\bar{Z}_\sigma\overset{(2.6)}{=}\tilde{\alpha}.\end{aligned}\tag{3.4}$$

From this it follows that

$$\begin{aligned}&\tilde{S}(S^{-1}(\bar{X}_\nu))\cdot\tilde{S}(S^{-1}(\beta))\cdot\tilde{S}(\bar{Y}_\nu)\tilde{\alpha}\bar{Z}_\nu\\&\quad\overset{(3.4)}{=}\tilde{S}(S^{-1}(\bar{X}_\nu))\cdot\tilde{S}(S^{-1}(\beta))\tilde{S}(\bar{Y}_\nu)\cdot v\alpha\bar{Z}_\nu\\&\quad\overset{(3.3)}{=}v\cdot S(S^{-1}(\bar{X}_\nu))\cdot S(S^{-1}(\beta))\cdot S(\bar{Y}_\nu)\alpha\bar{Z}_\nu\\&\quad=v\cdot\bar{X}_\nu\beta S(\bar{Y}_\nu)\alpha\bar{Z}_\nu\overset{2.6}{=}v,\end{aligned}$$

which proves (3.2) (i). To see v is invertible observe that

$$\begin{aligned}v\cdot S(X_\nu)\alpha Y_\nu\tilde{\beta}\tilde{S}(Z_\nu)&\overset{(3.3)}{=}\tilde{S}(X_\nu)v\alpha Y_\nu\tilde{\beta}\tilde{S}(Z_\nu)\\&\overset{(3.4)}{=}\tilde{S}(X_\nu)\tilde{\alpha}Y_\nu\tilde{\beta}\tilde{S}(Z_\nu)\\&\overset{(2.6)}{=}1,\end{aligned}$$

so

$$v^{-1}=S(X_\nu)\alpha Y_\nu\tilde{\beta}\tilde{S}(Z_\nu)$$

as stated.

Now using equation (2.10), we have

$$\begin{aligned}1\otimes v^{-1}&=\bar{X}_\nu\bar{X}_\mu X^{(1)}_\rho X_\sigma\otimes S\big(\bar{Y}^{(1)}_\nu\bar{Y}_\mu X^{(2)}_\rho Y_\sigma\big)\alpha\bar{Y}^{(2)}_\nu\bar{Z}_\mu Y_\rho\bar{Z}^{(1)}_\sigma\tilde{\beta}\tilde{S}\big(\bar{Z}_\nu Z_\rho Z^{(2)}_\sigma\big)\\&\overset{(2.7)}{=}\bar{X}_\mu X^{(1)}_\rho\otimes S\big(\bar{Y}_\mu X^{(2)}_\rho\big)\alpha\bar{Z}_\mu Y_\rho\tilde{\beta}\tilde{S}(Z_\rho).\end{aligned}$$

Applying $m\cdot(1\otimes\beta)$ gives

$$\begin{aligned}\beta v^{-1}&=\bar{X}_\mu X^{(1)}_\rho\beta S\big(\bar{Y}_\mu X^{(2)}_\rho\big)\alpha\bar{Z}_\mu Y_\rho\tilde{\beta}\tilde{S}(Z_\rho)\\&=\bar{X}_\mu\beta S(\bar{Y}_\mu)\alpha\bar{Z}_\mu\cdot\tilde{\beta}\overset{(2.6)}{=}\tilde{\beta},\end{aligned}\tag{3.5}$$

which completes the proof of (3.1). As to (3.2) (ii) observe that

$$\begin{aligned}&\tilde{X}_\nu\tilde{\beta}\tilde{S}(\bar{Y}_\nu)\tilde{S}(S^{-1}(\alpha))\tilde{S}(S^{-1}(\bar{Z}_\nu))\\&\quad\overset{(3.5)}{=}\bar{X}_\nu\beta v^{-1}\tilde{S}(\bar{Y}_\nu)\tilde{S}(S^{-1}(\alpha))\tilde{S}(S^{-1}(\bar{Z}_\nu))\\&\quad\overset{(3.3)}{=}\bar{X}_\nu\beta S(\bar{Y}_\nu)S(S^{-1}(\alpha))S(S^{-1}(\bar{Z}_\nu))v^{-1}\\&\quad=\bar{X}_\nu\beta S(\bar{Y}_\nu)\alpha\bar{Z}_\nu\cdot v^{-1}\overset{(2.6)}{=}v^{-1}\end{aligned}$$

as required. It finally remains to prove uniqueness. Hence, suppose $u \in H$ satisfies

$$uS(a) = \tilde{S}(a)u, \qquad \forall a \in H, \qquad u\alpha = \tilde{\alpha}, \qquad \tilde{\beta}u = \beta.$$

Then,

$$\begin{aligned} uv^{-1} &= u \cdot S(X_\nu)\alpha Y_\nu \tilde{\beta}\tilde{S}(Z_\nu) \\ &= \tilde{S}(X_\nu)u\alpha Y_\nu \tilde{\beta}\tilde{S}(Z_\nu) \\ &= \tilde{S}(X_\nu)\tilde{\alpha} Y_\nu \tilde{\beta}\tilde{S}(Z_\nu) \overset{(2.6)}{=} 1, \end{aligned}$$

which implies $u = v$ as required. □

In the special case $\tilde{S} = S$, we obtain the following useful result.

Corollary. *Suppose H is also a QHA with quasi-antipode $(S, \tilde{\alpha}, \tilde{\beta})$. Then there is a unique invertible central element $v \in H$, given explicitly by equation (3.2) (i) (with $\tilde{S} = S$), such that*

$$v\alpha = \tilde{\alpha}, \qquad \tilde{\beta}v = \beta.$$

It thus follows that the triple (S, α, β) satisfying (2.6), (2.7) for a QHA is not unique. Indeed following theorem 1, for arbitrary invertible $v \in H$, the triple $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$ defined by

$$\tilde{S}(a) = vS(a)v^{-1}, \qquad \forall a \in H; \qquad \tilde{\alpha} = v\alpha, \qquad \tilde{\beta} = \beta v^{-1}$$

is easily seen to satisfy (2.6), (2.7) and thus gives rise to a quasi-antipode $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$. Theorem 1 then shows that all such quasi-antipodes $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$ are obtainable this way; thus, there is a 1–1 correspondence between the latter and invertible $v \in H$. We say that these structures are *equivalent*, since they clearly give rise to equivalent QHA structures. Throughout we work with a fixed choice for the quasi-antipode (S, α, β).

We conclude this section with the following useful result, proved in [13], concerning the opposite QHA structure on H:

Proposition 1. *H is also a QHA, with co-unit ϵ, under the opposite co-product and co-associator $\Delta^T, \Phi^T \equiv \Phi_{321}^{-1}$, respectively, with quasi-antipode $(S^{-1}, \alpha^T = S^{-1}(\alpha), \beta^T = S^{-1}(\beta))$.*

The QHA $H^{\mathrm{cop}} \equiv (H, \Delta^T, \epsilon, \Phi^T, S^{-1}, \alpha^T, \beta^T)$ is called the opposite QHA structure. We remark that above we have adopted the notation of [12, 13] so that $\Delta^T = T \cdot \Delta$, where T is the usual twist map, and

$$\Phi_{321}^{-1} = \bar{Z}_\nu \otimes \bar{Y}_\nu \otimes \bar{X}_\nu.$$

This latter notation extends in a natural way and will be employed throughout.

4. Twisting

Let H be a quasi-bialgebra. Then $F \in H \otimes H$ is called a twist if it is invertible and satisfies the co-unit property

$$(\epsilon \otimes 1)F = (1 \otimes \epsilon)F = 1.$$

We recall that H is also a QBA with the same co-unit ϵ but with co-product and co-associator given by

$$\begin{aligned} &\Delta_F(a) = F\Delta(a)F^{-1}, \qquad \forall a \in H, \\ &\Phi_F = (F \otimes 1) \cdot (\Delta \otimes 1)F \cdot \Phi \cdot (1 \otimes \Delta)F^{-1} \cdot (1 \otimes F^{-1}), \end{aligned} \tag{4.1}$$

called the twisted structure induced by F. If moreover H is a QHA with quasi-antipode (S, α, β) then H is also a QHA under the above twisted structure with the *same* antipode S but with canonical elements

$$\alpha_F = m \cdot (1 \otimes \alpha)(S \otimes 1)F^{-1}, \qquad \beta_F = m \cdot (1 \otimes \beta)(1 \otimes S)F, \tag{4.2}$$

respectively. A detailed proof of these well-known results is given in [20]. We now investigate the behaviour of the operator v of theorem 1 under the twisted structure induced by F.

4.1. Universality of v

Recall that the operator v is given by

$$v = \tilde{S}(X_\nu)\tilde{\alpha} Y_\nu \beta S(Z_\nu).$$

Let $F \in H \otimes H$ be an arbitrary twist. We use the following notation for the twist F and its inverse F^{-1},

$$F = f_i \otimes f^i, \qquad F^{-1} = \bar{f}_i \otimes \bar{f}^i.$$

The twisted form of the co-associator is given by (4.1)

$$\Phi_F = X_\nu^F \otimes Y_\nu^F \otimes Z_\nu^F = f_i f_j^{(1)} X_\nu \bar{f}_k \otimes f^i f_j^{(2)} Y_\nu \bar{f}^k_{(1)} \bar{f}_l \otimes f^j Z_\nu \bar{f}^k_{(2)} \bar{f}^l. \tag{4.3}$$

For the twisted forms of the canonical elements we have from (4.2)

$$\begin{aligned} \tilde{\alpha}_F &= m \cdot (1 \otimes \tilde{\alpha})(\tilde{S} \otimes 1)F^{-1} = \tilde{S}(\bar{f}_p)\tilde{\alpha}\bar{f}^p, \\ \beta_F &= m \cdot (1 \otimes \beta)(1 \otimes S)F = f_q \beta S(f^q). \end{aligned} \tag{4.4}$$

We note that

$$\tilde{S}(f_j)\tilde{\alpha}_F f^j \overset{(4.4)}{=} \tilde{S}(\bar{f}_p f_j)\tilde{\alpha}\bar{f}^p f^j = m \cdot (1 \otimes \alpha)(\tilde{S} \otimes 1)(F^{-1}F) = \tilde{\alpha}, \tag{4.5}$$

and similarly,

$$\bar{f}_j \beta_F S(\bar{f}^j) = \beta. \tag{4.6}$$

The twisted form of v is given by

$$\begin{aligned} v_F &= \tilde{S}\big(X_\nu^F\big)\tilde{\alpha}_F Y_\nu^F \beta_F S\big(Z_\nu^F\big) \\ &\overset{(4.3)}{=} \tilde{S}\big(f_i f_j^{(1)} X_\nu \bar{f}_k\big)\tilde{\alpha}_F f^i f_j^{(2)} Y_\nu \bar{f}^k_{(1)} \bar{f}_l \beta_F S\big(f^j Z_\nu \bar{f}^k_{(2)} \bar{f}^l\big) \\ &= \tilde{S}\big(f_j^{(1)} X_\nu \bar{f}_k\big)\tilde{S}(f_i)\tilde{\alpha}_F f^i f_j^{(2)} Y_\nu \bar{f}^k_{(1)} \bar{f}_l \beta_F S(\bar{f}^l) S\big(f^j Z_\nu \bar{f}^k_{(2)}\big) \\ &\overset{(4.5)}{=} \tilde{S}\big(f_j^{(1)} X_\nu \bar{f}_k\big)\tilde{\alpha} f_j^{(2)} Y_\nu \bar{f}^k_{(1)} \bar{f}_l \beta_F S(\bar{f}^l) S\big(f^j Z_\nu \bar{f}^k_{(2)}\big) \\ &\overset{(4.6)}{=} \tilde{S}\big(f_j^{(1)} X_\nu \bar{f}_k\big)\tilde{\alpha} f_j^{(2)} Y_\nu \bar{f}^k_{(1)} \beta S\big(f^j Z_\nu \bar{f}^k_{(2)}\big) \\ &= \tilde{S}(X_\nu \bar{f}_k)\tilde{S}\big(f_j^{(1)}\big)\tilde{\alpha} f_j^{(2)} Y_\nu \bar{f}^k_{(1)} \beta S\big(\bar{f}^k_{(2)}\big) S(f^j Z_\nu) \\ &= \tilde{S}(X_\nu \bar{f}_k)\tilde{\alpha} Y_\nu \bar{f}^k_{(1)} \beta S\big(\bar{f}^k_{(2)}\big) S(Z_\nu) \\ &= \tilde{S}(X_\nu)\tilde{\alpha} Y_\nu \beta S(Z_\nu) = v, \end{aligned}$$

where in the last two lines we have used the antipode properties of α, β (2.7) and the co-unit property of twists. We have thus proved:

Theorem 2. *The operator v is universal (i.e., invariant under twisting).*

5. The Drinfeld twists

We turn our attention to the Drinfeld twist for the opposite structure of proposition 1. It is tempting to assume that F_δ^T qualifies as a Drinfeld twist for the opposite structure. However, this is not true since the antipode for the latter is S^{-1} rather than S. We shall show that the Drinfeld twist for the opposite structure is in fact related to the second Drinfeld twist which we define below. We begin with a review of the Drinfeld twist.

5.1. The Drinfeld twist

Observe that Δ' defined by

$$\Delta'(a) = (S \otimes S)\Delta^T(S^{-1}(a)), \qquad \forall a \in H \tag{5.1}$$

also determines a co-product on H. Associated with this co-product, we have a new QHA structure on H, which was proved in [13] and which we restate here:

Proposition 2. *H is also a QHA with the same co-unit ϵ and antipode S but with co-product Δ', co-associator $\Phi' = (S \otimes S \otimes S)\Phi_{321}$ and canonical elements $\alpha' = S(\beta)$, $\beta' = S(\alpha)$, respectively.*

Drinfeld has proved the remarkable result that this QHA structure is obtained by twisting with the Drinfeld twist, herein denoted as F_δ, given explicitly by

$$\begin{aligned}(\text{i}) \quad F_\delta &= (S \otimes S)\Delta^T(X_\nu) \cdot \gamma \cdot \Delta(Y_\nu \beta S(Z_\nu)),\\ &= \Delta'(\bar{X}_\nu \beta S(\bar{Y}_\nu)) \cdot \gamma \cdot \Delta(\bar{Z}_\nu),\end{aligned}$$

where

$$(\text{ii}) \quad \gamma = S(B_i)\alpha C_i \otimes S(A_i)\alpha D_i$$

with

$$(\text{iii}) \quad A_i \otimes B_i \otimes C_i \otimes D_i = \begin{cases} (\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1)\Phi \\ \text{or} \\ (1 \otimes \Phi) \cdot (1 \otimes 1 \otimes \Delta)\Phi^{-1}. \end{cases} \tag{5.2}$$

The inverse of F_δ is given explicitly by

$$\begin{aligned}(\text{i}) \quad F_\delta^{-1} &= \Delta(\bar{X}_\nu) \cdot \bar{\gamma} \cdot \Delta'(S(\bar{Y}_\nu)\alpha \bar{Z}_\nu)\\ &= \Delta(S(X_\nu)\alpha Y_\nu) \cdot \bar{\gamma} \cdot (S \otimes S)\Delta^T(Z_\nu),\end{aligned}$$

where

$$(\text{ii}) \quad \bar{\gamma} = \bar{A}_i \beta S(\bar{D}_i) \otimes \bar{B}_i \beta S(\bar{C}_i)$$

with

$$(\text{iii}) \quad \bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i = \begin{cases} (\Delta \otimes 1 \otimes 1)\Phi^{-1} \cdot (\Phi \otimes 1) \\ \text{or} \\ (1 \otimes 1 \otimes \Delta)\Phi \cdot (1 \otimes \Phi^{-1}). \end{cases} \tag{5.3}$$

The detailed proof that the QHA structure of proposition 2 is obtained by twisting with F_δ, as given in (5.2), and in particular

$$\Delta'(a) = F_\delta \Delta(a) F_\delta^{-1}, \qquad \forall a \in H \tag{5.4}$$

is proved in [13]. We simply state here some properties of γ, $\bar{\gamma}$ proved in [13] and which are crucial to the demonstration of Drinfeld's result:

Proposition 3.

$$\begin{aligned}
&\text{(i)} \quad (S\otimes S)\Delta^T(a_{(1)})\cdot\gamma\cdot\Delta(a_{(2)}) = \epsilon(a)\gamma, \qquad \forall a\in H,\\
&\text{(ii)} \quad \Delta(a_{(1)})\cdot\bar{\gamma}\cdot(S\otimes S)\Delta^T(a_{(2)}) = \epsilon(a)\bar{\gamma}, \qquad \forall a\in H, \qquad (5.5)\\
&\text{(iii)} \quad F_\delta\Delta(\alpha) = \gamma, \qquad \Delta(\beta)F_\delta^{-1} = \bar{\gamma}.
\end{aligned}$$

5.2. The second Drinfeld twist

Replacing S with S^{-1}, we obtain yet another co-product Δ_0 on H:

$$\Delta_0(a) = (S^{-1}\otimes S^{-1})\Delta^T(S(a)), \qquad \forall a\in H. \tag{5.1$'$}$$

We have the following analogue of proposition 2, the proof of which parallels that of [13] proposition 4, but with S and S^{-1} interchanged:

Proposition 2′. *H is also a QHA with the same co-unit ϵ and antipode S but with co-product Δ_0, co-associator $\Phi_0 = (S^{-1}\otimes S^{-1}\otimes S^{-1})\Phi_{321}$ and canonical elements $\alpha_0 = S^{-1}(\beta)$, $\beta_0 = S^{-1}(\alpha)$, respectively.*

By symmetry, we would expect this structure to be obtainable twisting. Indeed, we have

Theorem 3. *The QHA structure of proposition 2′ is obtained by twisting with*

$$F_0 \equiv (S^{-1}\otimes S^{-1})F_\delta^T \tag{5.6}$$

herein referred to as the second Drinfeld twist, where F_δ is the Drinfeld twist and $F_\delta^T = T\cdot F_\delta$.

Proof. It is clear that F_0 is invertible with inverse $F_0^{-1} = (S^{-1}\otimes S^{-1})\big(F_\delta^T\big)^{-1}$ and qualifies as a twist. For the co-product, we observe

$$\begin{aligned}
F_0\Delta(a)F_0^{-1} &= (S^{-1}\otimes S^{-1})F_\delta^T\cdot\Delta(a)\cdot(S^{-1}\otimes S^{-1})\big(F_\delta^T\big)^{-1}\\
&= (S^{-1}\otimes S^{-1})\cdot T\cdot\big[F_\delta^{-1}\cdot(S\otimes S)\Delta^T(a)\cdot F_\delta\big]\\
&= (S^{-1}\otimes S^{-1})\cdot T\cdot\big[F_\delta^{-1}\Delta'(S(a))F_\delta\big]\\
&\overset{(5.4)}{=} (S^{-1}\otimes S^{-1})\cdot T\cdot\Delta(S(a)) = (S^{-1}\otimes S^{-1})\Delta^T(S(a))\\
&\overset{(5.1')}{=} \Delta_0(a), \qquad \forall a\in H.
\end{aligned}$$

The co-associator is slightly more complicated though also simple. We have from Drinfeld's result

$$\Phi' \equiv (S\otimes S\otimes S)\Phi_{321} = (F_\delta\otimes 1)\cdot(\Delta\otimes 1)F_\delta\cdot\Phi\cdot(1\otimes\Delta)F_\delta^{-1}\cdot\big(1\otimes F_\delta^{-1}\big)$$

which implies

$$\begin{aligned}
(S\otimes S\otimes S)\Phi &= \big[(F_\delta\otimes 1)\cdot(\Delta\otimes 1)F_\delta\cdot\Phi\cdot(1\otimes\Delta)F_\delta^{-1}\cdot\big(1\otimes F_\delta^{-1}\big)\big]_{321}\\
&= \big(1\otimes F_\delta^T\big)\cdot(1\otimes\Delta^T)F_\delta^T\cdot\Phi_{321}\cdot\big(\Delta^T\otimes 1\big)F_\delta^{T\,-1}\cdot\big(F_\delta^{T\,-1}\otimes 1\big).
\end{aligned}$$

Applying $(S^{-1}\otimes S^{-1}\otimes S^{-1})$ gives

$$\begin{aligned}
\Phi &= \big(F_0^{-1}\otimes 1\big)\cdot(\Delta_0\otimes 1)F_0^{-1}\cdot\Phi_0\cdot(1\otimes\Delta_0)F_0\cdot(1\otimes F_0)\\
&= (\Delta\otimes 1)F_0^{-1}\cdot\big(F_0^{-1}\otimes 1\big)\cdot\Phi_0\cdot(1\otimes F_0)\cdot(1\otimes\Delta)F_0
\end{aligned}$$

with F_0 as in the theorem. Thus,

$$\Phi_0 = (F_0\otimes 1)\cdot(\Delta\otimes 1)F_0\cdot\Phi\cdot(1\otimes\Delta)F_0^{-1}\cdot\big(1\otimes F_0^{-1}\big),$$

which shows that indeed Φ_0 is obtained from Φ by twisting with F_0. The proof for the canonical elements is straightforward. □

5.3. The Drinfeld twists for the opposite structure

Recall that under the opposite structure of proposition 1, H is a QHA with antipode S^{-1}, co-product Δ^T and co-associator $\Phi^T = \Phi_{321}^{-1}$. It follows that if F_δ^0 is the Drinfeld twist for this opposite structure then, $\forall a \in H$,

$$\begin{aligned} F_\delta^0 \Delta^T(a)\big(F_\delta^0\big)^{-1} &= (\Delta^T)'(a) \\ &= (S^{-1}\otimes S^{-1})\Delta(S(a)) = \Delta_0^T(a) \end{aligned}$$

since S^{-1} is the antipode for this structure. On the other hand, if F_0 is the Drinfeld twist of equation (5.6), we also have

$$F_0^T \Delta^T(a)\big(F_0^T\big)^{-1} = \Delta_0^T(a)$$

with Δ_0 as in equation (5.1′). Here, we show in fact that $F_\delta^0 = F_0^T$.

Before proceeding we note that the Drinfeld twist is given by the canonical expression of equation (5.2) (i) with γ as in (5.2) (ii) constructed from the operator of (5.2) (iii); namely,

$$A_i \otimes B_i \otimes C_i \otimes D_i = \begin{cases} (\Phi^{-1}\otimes 1)\cdot(\Delta\otimes 1\otimes 1)\Phi \\ \text{or} \\ (1\otimes\Phi)\cdot(1\otimes 1\otimes\Delta)\Phi^{-1}. \end{cases}$$

This gives rise to two equivalent expansions for γ. Using the first expression we have, in obvious notation,

$$\begin{aligned} A_i \otimes B_i \otimes C_i \otimes D_i &= (\Phi^{-1}\otimes 1)\cdot(\Delta\otimes 1\otimes 1)\Phi \\ &= \bar{X}_\nu X_\mu^{(1)} \otimes \bar{Y}_\nu X_\mu^{(2)} \otimes \bar{Z}_\nu Y_\mu \otimes Z_\mu, \end{aligned}$$

which gives, upon substitution into (5.2) (ii),

$$\gamma = S\big(\bar{Y}_\nu X_\mu^{(2)}\big)\alpha \bar{Z}_\nu Y_\mu \otimes S\big(\bar{X}_\nu X_\mu^{(1)}\big)\alpha Z_\mu,$$

which is the expression obtained in [13]. On the other hand, using the second expression gives

$$\begin{aligned} A_i \otimes B_i \otimes C_i \otimes D_i &= (1\otimes\Phi)\cdot(1\otimes 1\otimes\Delta)\Phi^{-1} \\ &= \bar{X}_\mu \otimes X_\nu \bar{Y}_\mu \otimes Y_\nu \bar{Z}_\mu^{(1)} \otimes Z_\nu \bar{Z}_\mu^{(2)} \end{aligned}$$

and substituting into (5.2) (ii) gives the alternative expansion

$$\gamma = S(X_\nu \bar{Y}_\mu)\alpha Y_\nu \bar{Z}_\mu^{(1)} \otimes S(\bar{X}_\mu)\alpha Z_\nu \bar{Z}_\mu^{(2)} \tag{5.7}$$

which is equivalent to the expression above [13].

Using (5.2) (i) for the opposite structure, we have for the Drinfeld twist

$$F_\delta^0 = (S^{-1}\otimes S^{-1})\Delta\big(X_\nu^0\big)\cdot\gamma^0\cdot\Delta^T\big(Y_\nu^0 \beta^T S^{-1}\big(Z_\nu^0\big)\big),$$

where we have used the fact that the co-product for the opposite structure is Δ^T, the antipode is S^{-1}, with canonical elements $\alpha^T = S^{-1}(\alpha)$, $\beta^T = S^{-1}(\beta)$ and where we have set

$$X_\nu^0 \otimes Y_\nu^0 \otimes Z_\nu^0 = \Phi^T = \Phi_{321}^{-1},$$

which is the opposite co-associator, and where from (5.2) (ii)

$$\gamma^0 = S^{-1}\big(B_i^0\big)\alpha^T C_i^0 \otimes S^{-1}\big(A_i^0\big)\alpha^T D_i^0$$

with

$$\begin{aligned} A_i^0 \otimes B_i^0 \otimes C_i^0 \otimes D_i^0 &= [(\Phi^T)^{-1}\otimes 1]\cdot(\Delta^T\otimes 1\otimes 1)\Phi^T \\ &= (\Phi_{321}\otimes 1)\cdot(\Delta^T\otimes 1\otimes 1)\Phi_{321}^{-1}. \end{aligned}$$

In obvious notation, the latter is given by

$$(\Phi_{321}\otimes 1)\cdot(\Delta^T\otimes 1\otimes 1)\Phi_{321}^{-1} = Z_\nu \bar{Z}_\mu^{(2)} \otimes Y_\nu \bar{Z}_\mu^{(1)} \otimes X_\nu \bar{Y}_\mu \otimes \bar{X}_\mu$$

so that, using $\alpha^T = S^{-1}(\alpha)$,

$$\gamma^0 = S^{-1}\big(Y_\nu \bar{Z}_\mu^{(1)}\big) S^{-1}(\alpha) X_\nu \bar{Y}_\mu \otimes S^{-1}\big(Z_\nu \bar{Z}_\mu^{(2)}\big) S^{-1}(\alpha) \bar{X}_\mu \stackrel{(5.7)}{=} (S^{-1} \otimes S^{-1})(\gamma).$$

Thus we may write, using $\beta^T = S^{-1}(\beta)$,

$$F_\delta^0 = (S^{-1} \otimes S^{-1})\Delta\big(X_\nu^0\big) \cdot (S^{-1} \otimes S^{-1})\gamma \cdot \Delta^T\big(Y_\nu^0 S^{-1}(\beta) S^{-1}\big(Z_\nu^0\big)\big)$$

so that, substituting

$$X_\nu^0 \otimes Y_\nu^0 \otimes Z_\nu^0 = \Phi^T = \Phi_{321}^{-1} = \bar{Z}_\nu \otimes \bar{Y}_\nu \otimes \bar{X}_\nu,$$

gives

$$\begin{aligned} F_\delta^0 &= (S^{-1} \otimes S^{-1})\Delta(\bar{Z}_\nu) \cdot (S^{-1} \otimes S^{-1})\gamma \cdot \Delta^T(\bar{Y}_\nu S^{-1}(\beta) S^{-1}(\bar{X}_\nu)) \\ &= (S^{-1} \otimes S^{-1}) \cdot [(S \otimes S)\Delta^T(\bar{Y}_\nu S^{-1}(\bar{X}_\nu \beta)) \cdot \gamma \cdot \Delta(\bar{Z}_\nu)] \\ &= (S^{-1} \otimes S^{-1}) \cdot [\Delta'(\bar{X}_\nu \beta S(\bar{Y}_\nu)) \cdot \gamma \cdot \Delta(\bar{Z}_\nu)] \\ &\stackrel{(5.2)(i)}{=} (S^{-1} \otimes S^{-1}) F_\delta \stackrel{(5.6)}{=} F_0^T. \end{aligned}$$

Thus, we have proved

Proposition 4. *The Drinfeld twist for the opposite QHA structure of proposition 1 is given explicitly by*

$$F_\delta^0 = (S^{-1} \otimes S^{-1}) F_\delta = F_0^T.$$

To see how F_δ^T fits into the picture, we need to consider the second Drinfeld twist F_0 of theorem 3 associated with the co-product of equation (5.1′). We have immediately from proposition 4

Corollary. *The second Drinfeld twist for the opposite structure is* F_δ^T.

Proof. Since the antipode for the opposite structure is S^{-1}, theorem 3 implies that the second Drinfeld twist for this structure is $(S \otimes S)\big(F_\delta^0\big)^T$, where F_δ^0 is the Drinfeld twist for the opposite structure, given explicitly in proposition 4. It follows that the second Drinfeld twist for the opposite structure is

$$(S \otimes S) \cdot \big[(S^{-1} \otimes S^{-1}) F_\delta^T\big] = F_\delta^T. \qquad \square$$

5.4. Twisting the Drinfeld twist

It is first useful to determine the behaviour of $\bar{\gamma}$ in equation (5.3) (ii) under an arbitrary twist $G \in H \otimes H$. Under the twisted structure induced by G, the operator $\bar{\gamma}$ is twisted to $\bar{\gamma}_G$, given by equation (5.3) (ii, iii) for the twisted structure, so that

$$\text{(i)} \quad \bar{\gamma}_G = \bar{A}_i^G \beta_G S\big(\bar{D}_i^G\big) \otimes \bar{B}_i^G \beta_G S\big(\bar{C}_i^G\big)$$

where

$$\text{(ii)} \quad \bar{A}_i^G \otimes \bar{B}_i^G \otimes \bar{C}_i^G \otimes \bar{D}_i^G = (\Delta_G \otimes 1 \otimes 1)\Phi_G^{-1} \cdot (\Phi_G \otimes 1). \tag{5.8}$$

We have

Proposition 5. *Let $G = g_i \otimes g^i \in H \otimes H$ be a twist on a QHA H. Then,*

$$\bar{\gamma}_G = G \cdot \Delta(g_i) \cdot \bar{\gamma} \cdot (S \otimes S)(G^T \Delta^T(g^i)).$$

Proof. Throughout we write

$$G^{-1} = \bar{g}_i \otimes \bar{g}^i.$$

For the RHS of equation (5.8) (ii), we have

$$(\Delta_G \otimes 1 \otimes 1)\Phi_G^{-1} \cdot (\Phi_G \otimes 1) = (\Delta_G \otimes 1 \otimes 1) \cdot [(1 \otimes G) \cdot (1 \otimes \Delta)G \cdot \Phi^{-1} \cdot (\Delta \otimes 1)G^{-1} \cdot (G^{-1} \otimes 1)] \cdot \{[(G \otimes 1) \cdot (\Delta \otimes 1)G \cdot \Phi \cdot (1 \otimes \Delta)G^{-1} \cdot (1 \otimes G^{-1})] \otimes 1\},$$

where we have used equation (4.1) for Φ_G and its inverse, thus

$$\begin{aligned}
&(\Delta_G \otimes 1 \otimes 1)\Phi_G^{-1} \cdot (\Phi_G \otimes 1) \\
&\quad = (1 \otimes 1 \otimes G) \cdot (\Delta_G \otimes \Delta)G \cdot (\Delta_G \otimes 1 \otimes 1)\Phi^{-1} \\
&\qquad \cdot [(\Delta_G \otimes 1)\Delta \otimes 1]G^{-1} \cdot [(\Delta_G \otimes 1)G^{-1} \otimes 1] \cdot (G \otimes 1 \otimes 1) \cdot [(\Delta \otimes 1)G \otimes 1] \\
&\qquad \cdot (\Phi \otimes 1) \cdot [(1 \otimes \Delta)G^{-1} \otimes 1] \cdot (1 \otimes G^{-1} \otimes 1) \\
&\quad = (G \otimes G) \cdot (\Delta \otimes \Delta)G \cdot (\Delta \otimes 1 \otimes 1)\Phi^{-1} \cdot [(\Delta \otimes 1)\Delta \otimes 1]G^{-1} \\
&\qquad \cdot [(\Delta \otimes 1)G^{-1} \otimes 1] \cdot [(\Delta \otimes 1)G \otimes 1] \cdot (\Phi \otimes 1) \\
&\qquad \cdot [(1 \otimes \Delta)G^{-1} \otimes 1] \cdot (1 \otimes G^{-1} \otimes 1) \\
&\quad = (G \otimes G) \cdot (\Delta \otimes \Delta)G \cdot (\Delta \otimes 1 \otimes 1)\Phi^{-1} \cdot [(\Delta \otimes 1)\Delta \otimes 1]G^{-1} \cdot \\
&\qquad \cdot (\Phi \otimes 1) \cdot [(1 \otimes \Delta)G^{-1} \otimes 1] \cdot (1 \otimes G^{-1} \otimes 1) \\
&\quad \overset{(2.1)}{=} (G \otimes G) \cdot (\Delta \otimes \Delta)G \cdot \{(\Delta \otimes 1 \otimes 1)\Phi^{-1} \cdot (\Phi \otimes 1)\} \\
&\qquad \cdot [(1 \otimes \Delta)\Delta \otimes 1]G^{-1} \cdot [(1 \otimes \Delta)G^{-1} \otimes 1] \cdot (1 \otimes G^{-1} \otimes 1).
\end{aligned}$$

Now using the notation of equation (5.3) (iii), we have

$$(\Delta \otimes 1 \otimes 1)\Phi^{-1} \cdot (\Phi \otimes 1) = \bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i$$

so that in the notation of equation (5.8) (i)

$$\begin{aligned}
\bar{A}_i^G \otimes \bar{B}_i^G \otimes \bar{C}_i^G \otimes \bar{D}_i^G &= (\Delta_G \otimes 1 \otimes 1)\Phi_G^{-1} \cdot (\Phi_G \otimes 1) \\
&= (G \otimes G) \cdot (\Delta \otimes \Delta)G \cdot \{\bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i\} \cdot [(1 \otimes \Delta)\Delta \otimes 1]G^{-1} \\
&\quad \cdot [(1 \otimes \Delta)G^{-1} \otimes 1] \cdot (1 \otimes G^{-1} \otimes 1) \\
&= g_s g_j^{(1)} \bar{A}_i \bar{g}_l^{(1)} \bar{g}_k \otimes g^s g_j^{(2)} \bar{B}_i \bar{g}_{l(1)}^{(2)} \bar{g}_{(1)}^k \bar{g}_m \otimes g_t g_{(1)}^j \bar{C}_i \bar{g}_{l(2)}^{(2)} \bar{g}_{(2)}^k \bar{g}^m \otimes g^t g_{(2)}^j \bar{D}_i \bar{g}^l,
\end{aligned}$$

where we have used the obvious notation, so that

$$\Delta(g_i) = g_i^{(1)} \otimes g_i^{(2)}, \qquad (1 \otimes \Delta)\Delta(g_i) = g_i^{(1)} \otimes \Delta\big(g_i^{(2)}\big) = g_i^{(1)} \otimes g_{i(1)}^{(2)} \otimes g_{i(2)}^{(2)}, \quad \text{etc}$$

and all repeated indices are understood to be summed over. Substituting into equation (5.8) (i) gives

$$\begin{aligned}
\bar{\gamma}_G &= g_s g_j^{(1)} \bar{A}_i \bar{g}_l^{(1)} \bar{g}_k \beta_G S\big(g^t g_{(2)}^j \bar{D}_i \bar{g}^l\big) \otimes g^s g_j^{(2)} \bar{B}_i \bar{g}_{l(1)}^{(2)} \bar{g}_{(1)}^k \bar{g}_m \beta_G S\big(g_t g_{(1)}^j \bar{C}_i \bar{g}_{l(2)}^{(2)} \bar{g}_{(2)}^k \bar{g}^m\big) \\
&= g_s g_j^{(1)} \bar{A}_i \bar{g}_l^{(1)} \bar{g}_k \beta_G S\big(g^t g_{(2)}^j \bar{D}_i \bar{g}^l\big) \\
&\quad \otimes g^s g_j^{(2)} \bar{B}_i \bar{g}_{l(1)}^{(2)} \bar{g}_{(1)}^k \bar{g}_m \beta_G S(\bar{g}^m) S\big(\bar{g}_{(2)}^k\big) S\big(\bar{g}_{l(2)}^{(2)}\big) S\big(g_t g_{(1)}^j \bar{C}_i\big).
\end{aligned}$$

Now using

$$\bar{g}_m \beta_G S(\bar{g}^m) = (\beta_G)_{G^{-1}} = \beta_{G^{-1}G} = \beta \tag{5.9}$$

and making repeated use of equation (2.7) gives

$$\begin{aligned}
\bar{\gamma}_G &= g_s g_j^{(1)} \bar{A}_i \bar{g}_l^{(1)} \bar{g}_k \beta_G S\big(g^t g_{(2)}^j \bar{D}_i \bar{g}^l\big) \\
&\quad \otimes\ g^s g_j^{(2)} \bar{B}_i \bar{g}_{l(1)}^{(2)} \bar{g}_{(1)}^k \beta S\big(\bar{g}_{(2)}^k\big) S\big(\bar{g}_{l(2)}^{(2)}\big) S\big(g_t g_{(1)}^j \bar{C}_i\big) \\
&= g_s g_j^{(1)} \bar{A}_i \bar{g}_l \beta_G S(\bar{g}^l) S\big(g^t g_{(2)}^j \bar{D}_i\big) \otimes g^s g_j^{(2)} \bar{B}_i \beta S\big(g_t g_{(1)}^j \bar{C}_i\big)
\end{aligned}$$

$$\stackrel{(5.9)}{=} g_s g_j^{(1)} \bar{A}_i \beta S(\bar{D}_i) S\big(g^t g^j_{(2)}\big) \otimes g^s g_j^{(2)} \bar{B}_i \beta S(\bar{C}_i) S\big(g_t g^j_{(1)}\big)$$
$$\stackrel{(5.3)(\text{ii})}{=} \big(g_s g_j^{(1)} \otimes g^s g_j^{(2)}\big) \cdot \bar{\gamma} \cdot (S \otimes S)\big(g^t g^j_{(2)} \otimes g_t g^j_{(1)}\big)$$
$$= G \cdot \Delta(g_j) \cdot \bar{\gamma} \cdot (S \otimes S)(G^T \cdot \Delta^T(g^j))$$

which proves the result. □

We are now in a position to determine the action of an arbitrary twist $G \in H \otimes H$ on the inverse Drinfeld twist F_δ^{-1}, given in equation (5.3) (i). Under the twisted structure induced by G, F_δ^{-1} is twisted to $\big(F_\delta^G\big)^{-1} \equiv \big(F_\delta^{-1}\big)_G$, given as in equation (5.3) (i), but in terms of the twisted structure, so that, with the notation of equation (5.8), we have from (5.3) (i)

$$\big(F_\delta^G\big)^{-1} = \Delta_G\big(S\big(X_\nu^G\big)\alpha_G Y_\nu^G\big) \cdot \bar{\gamma}_G \cdot (S \otimes S)\Delta_G^T\big(Z_\nu^G\big)$$

with $\bar{\gamma}_G$ as in proposition 5.

In obvious notation, we may write

$$X_\nu^G \otimes Y_\nu^G \otimes Z_\nu^G = \Phi_G = (G \otimes 1) \cdot (\Delta \otimes 1)G \cdot \Phi \cdot (1 \otimes \Delta)G^{-1} \cdot (1 \otimes G^{-1})$$
$$= g_i g_j^{(1)} X_\nu \bar{g}_k \otimes g^i g_j^{(2)} Y_\nu \bar{g}^k_{(1)} \bar{g}_l \otimes g^j Z_\nu \bar{g}^k_{(2)} \bar{g}^l,$$

which implies

$$\big(F_\delta^G\big)^{-1} = \Delta_G\big[S\big(g_i g_j^{(1)} X_\nu \bar{g}_k\big)\alpha_G g^i g_j^{(2)} Y_\nu \bar{g}^k_{(1)} \bar{g}_l\big] \cdot \bar{\gamma}_G \cdot (S \otimes S)\Delta_G^T\big(g^j Z_\nu \bar{g}^k_{(2)} \bar{g}^l\big)$$
$$= \Delta_G\big[S(X_\nu \bar{g}_k) S\big(g_j^{(1)}\big) S(g_i)\alpha_G g^i g_j^{(2)} Y_\nu \bar{g}^k_{(1)} \bar{g}_l\big] \cdot \bar{\gamma}_G \cdot (S \otimes S)\Delta_G^T\big(g^j Z_\nu \bar{g}^k_{(2)} \bar{g}^l\big).$$

Using

$$S(g_i)\alpha_G g^i = (\alpha_G)_{G^{-1}} = \alpha_{G^{-1}G} = \alpha,$$

and equation (2.7), then gives

$$\big(F_\delta^G\big)^{-1} = \Delta_G\big[S(X_\nu \bar{g}_k)\alpha Y_\nu \bar{g}^k_{(1)} \bar{g}_l\big] \cdot \bar{\gamma}_G \cdot (S \otimes S)\Delta_G^T\big(Z_\nu \bar{g}^k_{(2)} \bar{g}^l\big)$$
$$= G \cdot \Delta\big[S(X_\nu \bar{g}_k)\alpha Y_\nu \bar{g}^k_{(1)} \bar{g}_l\big] \cdot G^{-1} \cdot \bar{\gamma}_G$$
$$\cdot (S \otimes S)(G^T)^{-1} \cdot (S \otimes S)\Delta^T\big(Z_\nu \bar{g}^k_{(2)} \bar{g}^l\big) \cdot (S \otimes S)G^T$$
$$\stackrel{\text{prop.}(5)}{=} G \cdot \Delta\big[S(X_\nu \bar{g}_k)\alpha Y_\nu \bar{g}^k_{(1)} \bar{g}_l\big] \cdot \Delta(g_i) \cdot \bar{\gamma}$$
$$\cdot (S \otimes S)\Delta^T(g^i) \cdot (S \otimes S)\Delta^T\big(Z_\nu \bar{g}^k_{(2)} \bar{g}^l\big) \cdot (S \otimes S)G^T$$
$$= G \cdot \Delta\big[S(X_\nu \bar{g}_k)\alpha Y_\nu \bar{g}^k_{(1)}\big] \cdot \Delta(\bar{g}_l)\Delta(g_i) \cdot \bar{\gamma}$$
$$\cdot (S \otimes S)\Delta^T(g^i) \cdot (S \otimes S)\Delta^T(\bar{g}^l) \cdot (S \otimes S)\Delta^T\big(Z_\nu \bar{g}^k_{(2)}\big) \cdot (S \otimes S)G^T$$
$$= G \cdot \Delta\big[S(X_\nu \bar{g}_k)\alpha Y_\nu \bar{g}^k_{(1)}\big] \cdot \Delta(\bar{g}_l g_i) \cdot \gamma$$
$$\cdot (S \otimes S)\Delta^T(\bar{g}^l g^i) \cdot (S \otimes S)\Delta^T(Z_\nu \bar{g}^k_{(2)}) \cdot (S \otimes S)G^T$$
$$= G \cdot \Delta[S(X_\nu \bar{g}_k)\alpha Y_\nu] \cdot \Delta\big(\bar{g}^k_{(1)}\big) \cdot \gamma$$
$$\cdot (S \otimes S)\Delta^T\big(\bar{g}^k_{(2)}\big) \cdot (S \otimes S)\Delta^T(Z_\nu) \cdot (S \otimes S)G^T,$$

where we have used the obvious result that

$$\bar{g}_l g_i \otimes \bar{g}^l g^i = G^{-1}G = 1 \otimes 1.$$

It then follows from proposition 3 that

$$\big(F_\delta^G\big)^{-1} = G \cdot \Delta[S(X_\nu)\alpha Y_\nu] \cdot \bar{\gamma} \cdot (S \otimes S)\Delta^T(Z_\nu) \cdot (S \otimes S)G^T$$
$$\stackrel{(5.3)(\text{i})}{=} G \cdot F_\delta^{-1} \cdot (S \otimes S)G^T.$$

We have thus proved

Theorem 4. *Let $G \in H \otimes H$ be a twist on a QHA H. Then under the twisted structure induced by G, F_δ^{-1} is twisted to*

$$\left(F_\delta^G\right)^{-1} \equiv \left(F_\delta^{-1}\right)_G = G \cdot F_\delta^{-1} \cdot (S \otimes S) G^T.$$

Equivalently, the Drinfeld twist is twisted to

$$F_\delta^G \equiv (F_\delta)_G = (S \otimes S)(G^T)^{-1} \cdot F_\delta \cdot G^{-1}.$$

Corollary. *F_0 as in equation (5.6) is twisted to*

$$F_0^G \equiv (F_0)_G = (S^{-1} \otimes S^{-1})(G^T)^{-1} \cdot F_0 \cdot G^{-1}.$$

Proof. Follows from the definition of $F_0 \equiv (S^{-1} \otimes S^{-1}) F_\delta^T$ and the theorem above. □

When H is quasi-triangular, the opposite structure of proposition 1 is obtainable, up to equivalence modulo (S, α, β), via twisting. In such a case, the results of section 3 have further useful consequences.

6. Quasi-triangular QHAs

A QHA H is called quasi-triangular if there exists an invertible element

$$\mathcal{R} = \sum_i e_i \otimes e^i \in H \otimes H$$

called the R-matrix, such that

$$\begin{aligned}
&\text{(i)} \quad \Delta^T(a)\mathcal{R} = \mathcal{R}\Delta(a), \qquad \forall a \in H, \\
&\text{(ii)} \quad (\Delta \otimes 1)\mathcal{R} = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}, \\
&\text{(iii)} \quad (1 \otimes \Delta)\mathcal{R} = \Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}\Phi_{123},
\end{aligned} \tag{6.1}$$

where

$$\mathcal{R}_{12} = e_i \otimes e^i \otimes 1, \qquad \mathcal{R}_{13} = e_i \otimes 1 \otimes e^i, \quad \text{etc.}$$

We first summarize some well-known results for quasi-triangular QHAs. It was shown in [13] that

Proposition 1′. *With the opposite QHA structure of proposition 1, H is also quasi-triangular with the R-matrix $\mathcal{R}^T = T \cdot \mathcal{R}$, called the opposite R-matrix.*

It follows from (6.1) (ii, iii) that

$$(\epsilon \otimes 1)\mathcal{R} = (1 \otimes \epsilon)\mathcal{R} = 1$$

so that $\mathcal{R}$ qualifies as a twist. Moreover, if $F \in H \otimes H$ is any twist then, as shown in [13], H is also quasi-triangular under the twisted structure of equations (4.1), (4.2) with the R-matrix

$$\mathcal{R}_F = F^T \mathcal{R} F^{-1}. \tag{6.2}$$

It was shown in [13] that

Proposition 6. *With the QHA structure of proposition 2, H is also quasi-triangular with the R-matrix*

$$\mathcal{R}' = (S \otimes S)\mathcal{R}.$$

We have seen that the QHA structure of proposition 2 is obtainable by twisting with the Drinfeld twist F_δ. It was further shown in [13] that the full structure of proposition 6 is also obtained by twisting with F_δ which, in view of equation (6.2), is equivalent to

$$(S\otimes S)\mathcal{R} = F_\delta^T \mathcal{R} F_\delta^{-1}. \tag{6.3}$$

This result in fact follows from the following relation:

$$(S\otimes S)\mathcal{R}\cdot\gamma = \gamma^T\mathcal{R},$$

where $\gamma^T = T\cdot\gamma$, proved in [13]. In view of proposition 3, this last equation is equivalent to

$$\mathcal{R}\bar{\gamma} = \bar{\gamma}^T\cdot(S\otimes S)\mathcal{R},$$

where $\bar{\gamma}^T = T\cdot\bar{\gamma}$, with γ and $\bar{\gamma}$ as in equations (5.2), (5.3).

In view of (6.1) (i), the opposite co-product is obtained from Δ by twisting with $\mathcal{R}$. In fact, we have the following result proved in [13]:

Proposition 7. *The opposite structure of propositions 1, 1′ is obtainable by twisting with the R-matrix $\mathcal{R}$ but with antipode S and canonical elements $\alpha_\mathcal{R}$, $\beta_\mathcal{R}$, respectively.*

Above $\alpha_\mathcal{R}$, $\beta_\mathcal{R}$ are given by equation (4.2), so that

$$\text{(i)}\quad \alpha_\mathcal{R} = m\cdot(1\otimes\alpha)(S\otimes 1)\mathcal{R}^{-1},\qquad \beta_\mathcal{R} = m\cdot(1\otimes\beta)(1\otimes S)\mathcal{R}.$$

Below we set

$$\text{(ii)}\quad \mathcal{R} = e_i\otimes e^i,\qquad \mathcal{R}^{-1} = \bar{e}_i\otimes\bar{e}^i$$

in terms of which we may write

$$\text{(iii)}\quad \alpha_\mathcal{R} = S(\bar{e}_i)\alpha\bar{e}^i,\qquad \beta_\mathcal{R} = e_i\beta S(e^i). \tag{6.4}$$

Thus with the co-product Δ^T and co-associator $\Phi^T = \Phi_{321}^{-1}$ of proposition 1, we have two QHA structures with differing quasi-antipodes $(S, \alpha_\mathcal{R}, \beta_\mathcal{R})$ and $(S^{-1}, \alpha^T, \beta^T)$ where, from proposition 1, $\alpha^T = S^{-1}(\alpha)$, $\beta^T = S^{-1}(\beta)$. It follows from theorem 1 that

Theorem 5. *There exists a unique invertible $u\in H$, such that*

$$S(a) = uS^{-1}(a)u^{-1}\qquad\text{or}\qquad S^2(a) = uau^{-1},\qquad \forall a\in H$$

and

$$uS^{-1}(\alpha) = \alpha_\mathcal{R},\qquad \beta_\mathcal{R}u = S^{-1}(\beta). \tag{6.5}$$

Explicitly,

$$\begin{aligned} u &= S(Y_\nu\beta S(Z_\nu))\alpha_\mathcal{R}X_\nu = S(\bar{Z}_\nu)\alpha_\mathcal{R}\bar{Y}_\nu S^{-1}(\beta)S^{-1}(\bar{X}_\nu)\\ u^{-1} &= Z_\nu\beta_\mathcal{R}S(S(X_\nu)\alpha Y_\nu) = S^{-1}(\bar{Z}_\nu)S^{-1}(\alpha)\bar{Y}_\nu\beta_\mathcal{R}S(\bar{X}_\nu). \end{aligned} \tag{6.6}$$

Above, we have used the fact that the opposite QHA structure has co-associator $\Phi^T = \Phi_{321}^{-1}$ and quasi-antipode $(S^{-1}, \alpha^T, \beta^T)$. We have then applied theorem 1 with $(\tilde{S}, \tilde{\alpha}, \tilde{\beta}) = (S, \alpha_\mathcal{R}, \beta_\mathcal{R})$ to give the result.

The above gives the u-operator of Drinfeld–Reshetikhin [5, 18]. It differs from, but is related to, the u-operator of Altschuler and Coste [1, 12]. To see how the latter arises, it is easily seen that $\tilde{\mathcal{R}}\equiv(\mathcal{R}^T)^{-1}$ also satisfies equation (6.1) and thus constitutes an R-matrix. Thus proposition 7 and theorem 3 also hold with $\mathcal{R}$ replaced by $\tilde{\mathcal{R}}$. This implies the existence of a unique invertible $\tilde{u}\in H$, such that

$$S^2(a) = \tilde{u}a\tilde{u}^{-1},\qquad \forall a\in H$$

and

$$\tilde{u}S^{-1}(\alpha) = \alpha_{\tilde{\mathcal{R}}}, \qquad \beta_{\tilde{\mathcal{R}}}\tilde{u} = S^{-1}(\beta)$$

with $\alpha_{\tilde{\mathcal{R}}}, \beta_{\tilde{\mathcal{R}}}$ as in equation (6.4) but with $\mathcal{R}$ replaced by $\tilde{\mathcal{R}}$. Explicitly we have, in this case,

$$\begin{aligned}\tilde{u} &= S(Y_\nu \beta S(Z_\nu))\alpha_{\tilde{\mathcal{R}}} X_\nu = S(\bar{Z}_\nu)\alpha_{\tilde{\mathcal{R}}}\bar{Y}_\nu S^{-1}(\beta)S^{-1}(\bar{X}_\nu)\\ \tilde{u}^{-1} &= Z_\nu \beta_{\tilde{\mathcal{R}}} S(S(X_\nu)\alpha Y_\nu) = S^{-1}(\bar{Z}_\nu)S^{-1}(\alpha)\bar{Y}_\nu \beta_{\tilde{\mathcal{R}}} S(\bar{X}_\nu).\end{aligned} \tag{6.7}$$

Then, as can be seen from [12] $\tilde{u}$ is precisely the u-operator of Altschuler and Coste.

To see the relation between u and $\tilde{u}$, we first note that $uS(u) = S(u)u$ is central. This follows by applying S to $S(a) = uS^{-1}(a)u^{-1}$, giving

$$S^2(a) = S(u^{-1})aS(u), \qquad \forall a \in H.$$

Before proceeding it is worth noting the following:

Lemma 1.

$$\begin{aligned} &\text{(i)} \quad \beta_{\tilde{\mathcal{R}}} = S(u)S(\beta), \qquad \alpha_{\tilde{\mathcal{R}}} = S(\alpha)S(u^{-1}),\\ &\text{(ii)} \quad \beta_{\mathcal{R}} = S(\tilde{u})S(\beta), \qquad \alpha_{\mathcal{R}} = S(\alpha)S(\tilde{u}^{-1}).\end{aligned} \tag{6.8}$$

Proof. By symmetry it suffices to prove (i). Now,

$$\begin{aligned}\beta_{\tilde{\mathcal{R}}} &= m\cdot(1\otimes\beta)(1\otimes S)(\mathcal{R}^T)^{-1} = \bar{e}^i\beta S(\bar{e}_i)\\ &\overset{(6.5)}{=} \bar{e}^i S(\beta_{\mathcal{R}}u)S(\bar{e}_i) = \bar{e}^i S(u)S(\beta_{\mathcal{R}})S(\bar{e}_i)\\ &= \bar{e}^i S(u)S[e_j\beta S(e^j)]S(\bar{e}_i)\\ &= \bar{e}^i S(u)S^2(e^j)S(\beta)S(e_j)S(\bar{e}_i)\\ &= S(u)S^2(\bar{e}^i)S^2(e^j)S(\beta)S(e_j)S(\bar{e}_i)\\ &= S(u)S^2(\bar{e}^i e^j)S(\beta)S(\bar{e}_i e_j) = S(u)S(\beta),\end{aligned}$$

where we have used the obvious result

$$\bar{e}_i e_j \otimes \bar{e}^i e^j = \mathcal{R}^{-1}\mathcal{R} = 1\otimes 1.$$

Similarly,

$$\begin{aligned}\alpha_{\tilde{\mathcal{R}}} &= m\cdot(1\otimes\alpha)(S\otimes 1)R^T = S(e^i)\alpha e_i\\ &\overset{(6.5)}{=} S(e^i)S(u^{-1}\alpha_{\mathcal{R}})e_i = S(e^i)S(\alpha_{\mathcal{R}})S(u^{-1})e_i\\ &= S(e^i)S[S(\bar{e}_j)\alpha\bar{e}^j]S(u^{-1})e_i\\ &= S(e^i)S(\bar{e}^j)S(\alpha)S^2(\bar{e}_j)S(u^{-1})e_i\\ &= S(e^i)S(\bar{e}^j)S(\alpha)S^2(\bar{e}_j)S^2(e_i)S(u^{-1})\\ &= S(\bar{e}^j e^i)S(\alpha)S^2(\bar{e}_j e_i)S(u^{-1}) = S(\alpha)S(u^{-1}).\end{aligned}$$

□

We are now in a position to prove

Lemma 2.

$$\tilde{u} = S(u^{-1})$$

Proof. From equation (6.7), we have

$$\begin{aligned}\tilde{u} &= S(Y_\nu\beta S(Z_\nu))\alpha_{\tilde{\mathcal{R}}}X_\nu\\ &\overset{(6.8)(\text{i})}{=} S(Y_\nu\beta S(Z_\nu))S(\alpha)S(u^{-1})X_\nu\\ &= S(Y_\nu\beta S(Z_\nu))S(\alpha)S^2(X_\nu)S(u^{-1})\\ &= S[S(X_\nu)\alpha Y_\nu\beta S(Z_\nu)]S(u^{-1})\\ &\overset{(2.6)}{=} S(u^{-1}).\end{aligned}$$

□

The above result clearly shows the connection between the u-operator of theorem 3 and that due to Altschuler and Coste. Obviously, the existence of the u-operator in the quasi-triangular case is a direct consequence of theorem 1 and proposition 7, the latter showing the equivalence of the opposite structure of proposition 1 with that due to twisting with $\mathcal{R}$. In the case H is not quasi-triangular, this opposite structure is not in general obtainable by a twist.

The operators u and $\tilde{u}$ are special cases of the v operator of theorem 1, it follows then from theorem 2 that

Theorem 6. *The operators u and $\tilde{u}$ are invariant under twisting.*

In section 3, we discussed the uniqueness of the quasi-antipode (S, α, β), but nothing has been said about the uniqueness of the twisted structures or the R-matrix in the quasi-triangular case. This is intimately connected with the quasi-cocycle condition to which we now turn.

7. The quasi-cocycle condition

The set of twists on a QHA H forms a group, moreover, the twisted structure of equations (4.1), (4.2) induced on a QHA H preserves this group structure in the following sense.

Lemma 3. *Let $F, G \in H \otimes H$ be twists on a QHA H. Then in the notation of equations (4.1), (4.2)*

$$\text{(i)} \quad \Delta_{FG} = (\Delta_G)_F, \qquad \Phi_{FG} = (\Phi_G)_F,$$
$$\text{(ii)} \quad \alpha_{FG} = (\alpha_G)_F, \qquad \beta_{FG} = (\beta_G)_F.$$

Moreover, if H is quasi-triangular then

$$\text{(iii)} \quad \mathcal{R}_{FG} = (\mathcal{R}_G)_F. \tag{7.1}$$

In other words, the structure obtained from twisting with G and then with F is the same as twisting with the twist FG. It is important that the right-hand side of equation (7.1) is interpreted correctly, e.g. $(\Phi_G)_F$ is given as in equation (4.1) but with Φ replaced by Φ_G and Δ by Δ_G, etc.

Given any QBA H, we may impose on a twist $F \in H \otimes H$ the following condition:

$$(F \otimes 1) \cdot (\Delta \otimes 1)F \cdot \Phi = \Phi \cdot (1 \otimes F) \cdot (1 \otimes \Delta)F \tag{7.2}$$

which we call the *quasi-cocycle condition*.

When $\Phi = 1 \otimes 1 \otimes 1$ this reduces to the usual cocycle condition on Hopf algebras. In the notation of equation (4.1), the quasi-cocycle condition is equivalent to

$$\Phi_F = \Phi. \tag{7.2$'$}$$

Thus twisting on a QBA by a twist F satisfying the quasi-cocycle condition results in a QBA structure with the same co-associator.

It is thus not surprising that the quasi-cocycle condition (7.2) is intimately related to the uniqueness of twisted structures on a QHA H. Indeed, if $F, G \in H \otimes H$ are twists giving rise to the *same* QBA structure, so that

$$\Delta_F = \Delta_G, \qquad \Phi_F = \Phi_G \tag{7.3}$$

then $C \equiv F^{-1}G$ must commute with the co-product Δ and satisfy the quasi-cocycle condition. Indeed in view of lemma 3, we have

$$\Delta_C = \Delta_{F^{-1}G} = (\Delta_G)_{F^{-1}} \overset{(7.3)}{=} (\Delta_F)_{F^{-1}} = \Delta_{F^{-1}F} = \Delta$$
$$\Phi_C = \Phi_{F^{-1}G} = (\Phi_G)_{F^{-1}} \overset{(7.3)}{=} (\Phi_F)_{F^{-1}} = \Phi_{F^{-1}F} = \Phi.$$

This leads to the following:

Definition 3. *A twist $C \in H \otimes H$ on any QBA H is called compatible if*

(i) Ccommutes with the co-product Δ,

(ii) Csatisfies the quasi-cocycle condition.

In other words, twisting a QBA H with a compatible twist C gives exactly the same QBA structure. The set of compatible twists on H thus forms a subgroup of the group of twists on H.

Proposition 8. *Let $F, G \in H \otimes H$ be twists on a QBA H. Then the twisted structures induced by F and G coincide if and only if there exists a compatible twist $C \in H \otimes H$, such that $G = FC$.*

Proof. We have already seen that if F, G give rise to the same QBA structure then $C = F^{-1}G$ is a compatible twist and $G = FC$. Conversely, suppose C is a compatible twist and set $G = FC$. Then,

$$\Delta_G = \Delta_{FC} = (\Delta_C)_F = \Delta_F$$
$$\Phi_G = \Phi_{FC} = (\Phi_C)_F = \Phi_F,$$

so that G gives precisely the same twisted structure as F. □

Setting $G = 1 \otimes 1$ into the above gives

Corollary. *Let $F \in H \otimes H$ be a twist on a QBA H. Then the twisted structure induced by F coincides with the structure on H if and only if F is a compatible twist.*

In view of the group properties of twists, the above corollary is equivalent to proposition 8.

Let H be a quasi-triangular QHA with the R-matrix $\mathcal{R}$ satisfying equation (6.1). From proposition 7, the opposite co-associator $\Phi^T = \Phi_{321}^{-1}$ and co-product Δ^T are obtained by twisting with $\mathcal{R}$, so that $\Phi^T = \Phi_{\mathcal{R}}$. The proof of this result utilizes only the properties (6.1). Hence, since

$$\Phi = \Phi_{\mathcal{R}^{-1}\mathcal{R}} = (\Phi_{\mathcal{R}})_{\mathcal{R}^{-1}} = (\Phi^T)_{\mathcal{R}^{-1}}$$

it follows that if Q is another R-matrix for H, i.e. satisfies equation (6.1), then we must have also

$$(\Phi^T)_{Q^{-1}} = \Phi.$$

Then $Q^{-1}\mathcal{R}$ must qualify as a compatible twist. Indeed, it obviously commutes with Δ, while as to the quasi-cocycle condition, we have

$$\Phi_{Q^{-1}\mathcal{R}} = (\Phi_{\mathcal{R}})_{Q^{-1}} = (\Phi^T)_{Q^{-1}} = \Phi.$$

Note that $(Q^T)^{-1}, (\mathcal{R}^T)^{-1}$ also determine R-matrices so the following must all determine compatible twists: $Q^{-1}\mathcal{R}, Q^T\mathcal{R}, \mathcal{R}^{-1}Q, \mathcal{R}^TQ$. In particular $\mathcal{R}^T\mathcal{R}$ must determine a compatible twist, as may be verified directly.

With the notation of section 4, it is easily seen that the operator

$$A = \Delta(u^{-1})F_\delta^{-1}(u \otimes u)F_0 = F_\delta^{-1}(u \otimes u)F_0\Delta(u^{-1}) \tag{7.4}$$

commutes with Δ. This operator appears in the work of Altschuler and Coste [1] in connection with ribbon QHAs. The operator A satisfies the quasi-cocycle condition and thus determines a compatible twist.

For general QBAs H, to see that there are sufficiently many compatible twists, we have

Lemma 4. *Let $z \in H$ be an invertible central element. Then,*

$$C = (z \otimes z)\Delta(z^{-1})$$

is a compatible twist.

Proof. Obviously, C commutes with the co-product Δ so it remains to prove that it satisfies the quasi-cocycle condition. To this end note that

$$\begin{aligned}(C \otimes 1)(\Delta \otimes 1)C &= (z \otimes z \otimes 1)(\Delta(z^{-1}) \otimes 1)(\Delta(z) \otimes z)(\Delta \otimes 1)\Delta(z^{-1}) \\ &= (z \otimes z \otimes z)(\Delta \otimes 1)\Delta(z^{-1}) \qquad (7.5)\end{aligned}$$

and similarly

$$\begin{aligned}(1 \otimes C)(1 \otimes \Delta)C &= (1 \otimes z \otimes z)(1 \otimes \Delta(z^{-1}))(z \otimes \Delta(z))(1 \otimes \Delta)\Delta(z^{-1}) \\ &= (z \otimes z \otimes z)(1 \otimes \Delta)\Delta(z^{-1}) \qquad (7.6)\end{aligned}$$

thus

$$\begin{aligned}(C \otimes 1)(\Delta \otimes 1)C\Phi &\overset{(7.5)}{=} (z \otimes z \otimes z)(\Delta \otimes 1)\Delta(z^{-1})\Phi \\ &\overset{(2.1)}{=} (z \otimes z \otimes z)\Phi(1 \otimes \Delta)\Delta(z^{-1}) \\ &\overset{(7.6)}{=} (z \otimes z \otimes z)\Phi(z^{-1} \otimes z^{-1} \otimes z^{-1})(1 \otimes C)(1 \otimes \Delta)C \\ &= \Phi(1 \otimes C)(1 \otimes \Delta)C.\end{aligned}$$

□

With C as in the lemma, we see that

$$(\epsilon \otimes 1)C = (1 \otimes \epsilon)C = \epsilon(z).$$

Thus, strictly speaking, $\epsilon(z^{-1})C$ qualifies as a compatible twist.

Following Altschuler and Coste [1], a quasi-triangular QHA is called a ribbon QHA if the operator A of equation (7.4) is given by

$$A = (v \otimes v)\Delta(v^{-1})$$

for a certain invertible central element v, related to the u-operator u. This is consistent with the lemma above and the fact that A determines a compatible twist.

In the case of ribbon Hopf algebras, we have $\mathcal{R}^T\mathcal{R} = (v \otimes v)\Delta(v^{-1})$, so that the compatible twist $\mathcal{R}^T\mathcal{R}$ is also of the form of lemma 4. This may not be the case for quasi-triangular QHAs in general.

It is worth noting that if H is a QHA and $C \in H \otimes H$ a compatible twist then H is also a QHA under the twisted structure induced by C with exactly the same co-product Δ, co-unit ϵ, co-associator Φ, antipode S, but with canonical elements given by equation (4.2); namely,

$$\alpha_C = m \cdot (S \otimes 1)(1 \otimes \alpha)C^{-1}, \qquad \beta_C = m \cdot (1 \otimes S)(1 \otimes \beta)C.$$

In view of theorem 1 and its corollary, we have immediately

Proposition 9. *Suppose $C \in H \otimes H$ is a compatible twist on a QHA H. Then there exists a unique invertible central element $z \in H$, such that*

$$z\alpha = \alpha_C, \qquad \beta_C z = \beta.$$

Explicitly

$$\begin{aligned}z &= S(X_\nu)\alpha_C Y_\nu \beta S(Z_\nu) = \bar{X}_\nu \beta S(\bar{Y}_\nu)\alpha_C \bar{Z}_\nu \\ z^{-1} &= S(X_\nu)\alpha Y_\nu \beta_C S(Z_\nu) = \bar{X}_\nu \beta_C S(\bar{Y}_\nu)\alpha \bar{Z}_\nu.\end{aligned}$$

In the case H is quasi-triangular, we have seen that $C = \mathcal{R}^T\mathcal{R}$ is a compatible twist. Since the latter form a group, we have the infinite family of compatible twists $C = (\mathcal{R}^T\mathcal{R})^m$, $m \in \mathbb{Z}$, in which case the central elements $z^{\pm 1}$ of proposition 9 give the quadratic invariants of [12].

We conclude this section by noting, in the quasi-triangular case, that twisting the Drinfeld twist with the R-matrix $\mathcal{R}$ gives, from theorem 4, the twisted Drinfeld twist

$$F_\delta^{\mathcal{R}} \equiv (F_\delta)_{\mathcal{R}} = (S \otimes S)(\mathcal{R}^T)^{-1} \cdot F_\delta \cdot \mathcal{R}^{-1}.$$

On the other hand, since $(\mathcal{R}^T)^{-1}$ is an R-matrix we have, from equation (6.3),

$$(S \otimes S)(\mathcal{R}^T)^{-1} = F_\delta^T(\mathcal{R}^T)^{-1}F_\delta^{-1}$$

which implies

$$F_\delta^{\mathcal{R}} = F_\delta^T(\mathcal{R}^T)^{-1} \cdot \mathcal{R}^{-1} = F_\delta^T(\mathcal{R}\mathcal{R}^T)^{-1}$$

where $\mathcal{R}\mathcal{R}^T$ and its inverse are compatible twists under the opposite structure. This shows that F_δ^T will give rise to a Drinfeld twist under the opposite structure of proposition 7 induced by twisting with $\mathcal{R}$ (which has antipode S rather than S^{-1}). Applying T to the equation above gives

$$\left(F_\delta^R\right)^T = F_\delta(\mathcal{R}^T\mathcal{R})^{-1}$$

which shows that, since $\mathcal{R}^T\mathcal{R}$ and its inverse are compatible twists, $\left(F_\delta^{\mathcal{R}}\right)^T$ also gives rise to a Drinfeld twist on H.

8. Quasi-dynamical QYBE

Throughout we assume H is a quasi-triangular QHA with the R-matrix $\mathcal{R}$ satisfying (6.1) which we reproduce here:

$$\begin{aligned} &\text{(i)} && \Delta^T(a)\mathcal{R} = \mathcal{R}\Delta(a), \qquad \forall a \in H,\\ &\text{(ii)} && (\Delta \otimes 1)\mathcal{R} = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1},\\ &\text{(iii)} && (1 \otimes \Delta)\mathcal{R} = \Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}\Phi_{123}. \end{aligned} \tag{6.1$'$}$$

Applying $T \otimes 1$ to (ii) and $1 \otimes T$ to (iii) then gives

$$\begin{aligned} &\text{(ii}'\text{)} && (\Delta^T \otimes 1)\mathcal{R} = \Phi_{321}^{-1}\mathcal{R}_{23}\Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1},\\ &\text{(iii}'\text{)} && (1 \otimes \Delta^T)\mathcal{R} = \Phi_{321}\mathcal{R}_{12}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}. \end{aligned}$$

It follows that

$$\mathcal{R}_{12}(\Delta \otimes 1)\mathcal{R} = (\Delta^T \otimes 1)\mathcal{R} \cdot \mathcal{R}_{12}$$

from which we deduce that $\mathcal{R}$ must satisfy the quasi-QYBE:

$$\mathcal{R}_{12}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1} = \Phi_{321}^{-1}\mathcal{R}_{23}\Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}. \tag{8.1}$$

If we twist H with a twist $F \in H \otimes H$ then H is also a quasi-triangular QHA under the twisted structure (4.1), (4.2) induced by F with the universal R-matrix

$$\mathcal{R}_F = F^T\mathcal{R}F^{-1}.$$

Following equation (7.2), we say a twist $F(\lambda) \in H \otimes H$ satisfies the shifted quasi-cocycle condition if

$$[F(\lambda) \otimes 1] \cdot (\Delta \otimes 1)F(\lambda) \cdot \Phi = \Phi \cdot [1 \otimes F(\lambda + h^{(1)})] \cdot (1 \otimes \Delta)F(\lambda), \tag{8.2}$$

where $\lambda \in H$ depends on one (or possibly several) parameters and $h \in H$ is fixed. Alternatively, we may write in obvious notation

$$F_{12}(\lambda) \cdot (\Delta \otimes 1)F(\lambda) \cdot \Phi = \Phi \cdot F_{23}(\lambda + h^{(1)}) \cdot (1 \otimes \Delta)F(\lambda). \tag{8.2$'$}$$

When $h = 0$, this reduces to the quasi-cocycle condition (7.2) satisfied by $F = F(\lambda)$. When $\Phi = 1 \otimes 1 \otimes 1$ (i.e., the normal Hopf-algebra case) equation (8.2) reduces to the usual shifted cocycle condition.

Twisting H with a twist F satisfying the (unshifted) quasi-cocycle condition results in a QHA with the same co-associator Φ, co-unit ϵ and antipode S but with the twisted co-product Δ_F, R-matrix $\mathcal{R}_F$ (and canonical elements α_F, β_F). We now consider twisting H with a twist $F = F(\lambda)$ satisfying the shifted condition (8.2). Then under this twisted structure H is also a quasi-triangular QHA with the same co-unit ϵ and antipode S but with the co-associator $\Phi(\lambda) = \Phi_{F(\lambda)}$, and the co-product and the R-matrix given by

$$\Delta_\lambda(a) = F(\lambda)\Delta(a)F(\lambda)^{-1}, \qquad \forall a \in H, \qquad \mathcal{R}(\lambda) = F^T(\lambda)\mathcal{R}F(\lambda)^{-1} \tag{8.3}$$

with canonical elements $\alpha_\lambda = \alpha_{F(\lambda)}$, $\beta_\lambda = \beta_{F(\lambda)}$.

In view of equation (8.2′), we have for the co-associator

$$\begin{aligned}\Phi(\lambda) &= F_{12}(\lambda) \cdot (\Delta \otimes 1)F(\lambda) \cdot \Phi \cdot (1 \otimes \Delta)F(\lambda)^{-1} \cdot F_{23}(\lambda)^{-1} \\ &= \Phi \cdot F_{23}(\lambda + h^{(1)}) \cdot (1 \otimes \Delta)F(\lambda) \cdot (1 \otimes \Delta)F(\lambda)^{-1} \cdot F_{23}(\lambda)^{-1} \\ &= \Phi \cdot F_{23}(\lambda + h^{(1)}) \cdot F_{23}(\lambda)^{-1}\end{aligned} \tag{8.4}$$

which implies

$$\Phi(\lambda)^{-1} = F_{23}(\lambda) \cdot F_{23}(\lambda + h^{(1)})^{-1} \cdot \Phi^{-1}.$$

In the Hopf-algebra case, equation (8.4) reduces to the expression for $\Phi(\lambda)$ obtained in [13] $(\Phi = 1 \otimes 1 \otimes 1)$.

Under the above twisted structure equation (6.1) (ii) becomes

$$(\Delta_\lambda \otimes 1)\mathcal{R}(\lambda) = \Phi_{231}(\lambda)^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132}(\lambda) \cdot \mathcal{R}_{23}(\lambda) \cdot \Phi_{123}^{-1}(\lambda).$$

Now

$$\begin{aligned}\Phi_{132}(\lambda) &= (1 \otimes T)\Phi_{123}(\lambda) \\ &\overset{(8.4)}{=} \Phi_{132} \cdot F_{23}^T(\lambda + h^{(1)}) \cdot F_{23}^T(\lambda)^{-1}\end{aligned} \tag{8.5}$$

which implies

$$\begin{aligned}(\Delta_\lambda \otimes 1)\mathcal{R}(\lambda) &= \Phi_{231}(\lambda)^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132} \cdot F_{23}^T(\lambda + h^{(1)}) \cdot F_{23}^T(\lambda)^{-1} \cdot \mathcal{R}_{23}(\lambda) \cdot \Phi_{123}^{-1}(\lambda) \\ &\overset{(8.4)}{=} \Phi_{231}(\lambda)^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132} \cdot F_{23}^T(\lambda + h^{(1)}) \\ &\quad \cdot F_{23}^T(\lambda)^{-1} \cdot \mathcal{R}_{23}(\lambda) \cdot F_{23}(\lambda) \cdot F_{23}(\lambda + h^{(1)})^{-1} \cdot \Phi_{123}^{-1} \\ &\overset{(8.3)}{=} \Phi_{231}(\lambda)^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132} \cdot \mathcal{R}_{23}(\lambda + h^{(1)}) \cdot \Phi_{123}^{-1}.\end{aligned}$$

Similarly equation (6.1) (iii) becomes

$$(1 \otimes \Delta_\lambda)\mathcal{R}(\lambda) = \Phi_{312}(\lambda) \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{213}^{-1}(\lambda) \cdot \mathcal{R}_{12}(\lambda) \cdot \Phi_{123}(\lambda).$$

Now

$$\begin{aligned}\Phi_{312}(\lambda) &= (T \otimes 1)(1 \otimes T)\Phi_{123}(\lambda) \\ &\overset{(8.4)}{=} (T \otimes 1)\left[\Phi_{132} \cdot F_{23}^T(\lambda + h^{(1)}) \cdot F_{23}^T(\lambda)^{-1}\right] \\ &= \Phi_{312} \cdot F_{13}^T(\lambda + h^{(2)}) \cdot F_{13}^T(\lambda)^{-1}\end{aligned}$$

while

$$\begin{aligned}\Phi_{213}^{-1}(\lambda) &= (T \otimes 1)\Phi(\lambda)^{-1} \\ &\overset{(8.4)}{=} (T \otimes 1)[F_{23}(\lambda) \cdot F_{23}(\lambda + h^{(1)})^{-1} \cdot \Phi^{-1}] \\ &= F_{13}(\lambda) \cdot F_{13}(\lambda + h^{(2)})^{-1} \cdot \Phi_{213}^{-1}.\end{aligned}$$

Therefore,

$$(1\otimes\Delta_\lambda)\mathcal{R}(\lambda)=\Phi_{312}\cdot F_{13}^T(\lambda+h^{(2)})\cdot F_{13}^T(\lambda)^{-1}\cdot\mathcal{R}_{13}(\lambda)\cdot F_{13}(\lambda)\cdot F_{13}(\lambda+h^{(2)})^{-1}\cdot\Phi_{213}^{-1}\cdot\mathcal{R}_{12}(\lambda)\cdot\Phi_{123}(\lambda)$$
$$\overset{(8.3)}{=}\Phi_{312}\cdot R_{13}(\lambda+h^{(2)})\cdot\Phi_{213}^{-1}\cdot R_{12}(\lambda)\cdot\Phi_{123}(\lambda).$$

We thus arrive at

Lemma 5. *$\mathcal{R}(\lambda)$ satisfies the co-product properties*

$$\begin{aligned}
&\text{(i)} && (\Delta_\lambda\otimes 1)\mathcal{R}(\lambda)=\Phi_{231}^{-1}(\lambda)\cdot\mathcal{R}_{13}(\lambda)\cdot\Phi_{132}\cdot\mathcal{R}_{23}(\lambda+h^{(1)})\cdot\Phi_{123}^{-1},\\
&\text{(ii)} && (1\otimes\Delta_\lambda)\mathcal{R}(\lambda)=\Phi_{312}\cdot R_{13}(\lambda+h^{(2)})\cdot\Phi_{213}^{-1}\cdot R_{12}(\lambda)\cdot\Phi_{123}(\lambda),\\
&\text{(iii)} && \left(\Delta_\lambda^T\otimes 1\right)\mathcal{R}(\lambda)=\Phi_{321}^{-1}(\lambda)\cdot\mathcal{R}_{23}(\lambda)\cdot\Phi_{312}\cdot\mathcal{R}_{13}(\lambda+h^{(2)})\cdot\Phi_{213}^{-1},\\
&\text{(iv)} && \left(1\otimes\Delta_\lambda^T\right)\mathcal{R}(\lambda)=\Phi_{321}\cdot R_{12}(\lambda+h^{(3)})\cdot\Phi_{231}^{-1}\cdot R_{13}(\lambda)\cdot\Phi_{132}(\lambda).
\end{aligned}\tag{8.6}$$

Proof. We have already proved (i) and (ii) while (iii) follows by applying $(T\otimes 1)$ to (i) and (iv) by applying $(1\otimes T)$ to (ii). □

We are now in a position to determine the QQYBE (8.1) satisfied by $\mathcal{R}=\mathcal{R}(\lambda)$ for this twisted structure. We have

$$\begin{aligned}
&\mathcal{R}_{23}(\lambda)\cdot\Phi_{312}\cdot\mathcal{R}_{13}(\lambda+h^{(2)})\cdot\Phi_{213}^{-1}\cdot R_{12}(\lambda)\\
&\overset{(8.6)(\text{ii})}{=}\mathcal{R}_{23}(\lambda)\cdot(1\otimes\Delta_\lambda)\mathcal{R}(\lambda)\cdot\Phi_{123}^{-1}(\lambda)\\
&\overset{(6.1)(\text{i})}{=}\left(1\otimes\Delta_\lambda^T\right)\mathcal{R}(\lambda)\cdot\mathcal{R}_{23}(\lambda)\cdot\Phi_{123}^{-1}(\lambda)\\
&\overset{(8.6)(\text{iv})}{=}\Phi_{321}\cdot R_{12}(\lambda+h^{(3)})\cdot\Phi_{231}^{-1}\cdot R_{13}(\lambda)\cdot\Phi_{132}(\lambda)\cdot\mathcal{R}_{23}(\lambda)\cdot\Phi_{123}^{-1}(\lambda)
\end{aligned}$$

where for the last three terms we have

$$\begin{aligned}
\Phi_{132}(\lambda)\cdot\mathcal{R}_{23}(\lambda)\cdot\Phi_{123}^{-1}(\lambda)&\overset{(8.4,8.5)}{=}\Phi_{132}\cdot F_{23}^T(\lambda+h^{(1)})\cdot F_{23}^T(\lambda)^{-1}\\
&\quad\cdot R_{23}(\lambda)\cdot F_{23}(\lambda)\cdot F_{23}(\lambda+h^{(1)})^{-1}\cdot\Phi_{123}^{-1}\\
&\overset{(8.3)}{=}\Phi_{132}\cdot\mathcal{R}_{23}(\lambda+h^{(1)})\cdot\Phi_{123}^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathcal{R}_{23}(\lambda)\cdot\Phi_{312}\cdot\mathcal{R}_{13}(\lambda+h^{(2)})\cdot\Phi_{213}^{-1}\cdot R_{12}(\lambda)\\
&\quad=\Phi_{321}\cdot\mathcal{R}_{12}(\lambda+h^{(3)})\cdot\Phi_{231}^{-1}\cdot\mathcal{R}_{13}(\lambda)\cdot\Phi_{132}\cdot\mathcal{R}_{23}(\lambda+h^{(1)})\cdot\Phi_{123}^{-1}.
\end{aligned}$$

We thus arrive at

Proposition 10. *$\mathcal{R}(\lambda)$ satisfies the quasi-dynamical QYBE*

$$\begin{aligned}
&\mathcal{R}_{12}(\lambda+h^{(3)})\cdot\Phi_{231}^{-1}\cdot\mathcal{R}_{13}(\lambda)\cdot\Phi_{132}\cdot\mathcal{R}_{23}(\lambda+h^{(1)})\cdot\Phi_{123}^{-1}\\
&\quad=\Phi_{321}^{-1}\cdot\mathcal{R}_{23}(\lambda)\cdot\Phi_{312}\cdot\mathcal{R}_{13}(\lambda+h^{(2)})\cdot\Phi_{213}^{-1}\cdot\mathcal{R}_{12}(\lambda).
\end{aligned}\tag{8.7}$$

In the Hopf algebra case ($\Phi=1\otimes 1\otimes 1$), equation (8.7) reduces to the usual dynamical QYBE. If we set $h=0$, then equation (8.7) reduces to the quasi-QYBE (8.1) satisfied by $\mathcal{R}=\mathcal{R}(\lambda)$. Hence, the term quasi-dynamical QYBE for (8.7): we could, alternatively, refer to (8.7) as the dynamical quasi-QYBE (dynamical QQYBE), since it is obviously the quasi-Hopf algebra analogue of the usual dynamical QYBE.

With respect to the QHA structure of propositions 2, 2′, we have the R-matrices

$$\mathcal{R}'(\lambda)=(S\otimes S)\mathcal{R}(\lambda),\qquad\mathcal{R}_0(\lambda)=(S^{-1}\otimes S^{-1})\mathcal{R}(\lambda),$$

respectively. Then applying $(S \otimes S \otimes S)$, $(S^{-1} \otimes S^{-1} \otimes S^{-1})$ respectively to equation (8.7), it follows that both of these R-matrices satisfy the opposite quasi-dynamical QYBE

$$\tilde{\mathcal{R}}_{12}(\lambda) \cdot \tilde{\Phi}_{231}^{-1} \cdot \tilde{\mathcal{R}}_{13}(\lambda + h^{(2)}) \cdot \tilde{\Phi}_{132} \cdot \tilde{\mathcal{R}}_{23}(\lambda) \cdot \tilde{\Phi}_{123}^{-1} = \tilde{\Phi}_{321}^{-1} \cdot \tilde{\mathcal{R}}_{23}(\lambda + h^{(1)}) \cdot \tilde{\Phi}_{312} \cdot \tilde{\mathcal{R}}_{13}(\lambda) \cdot \tilde{\Phi}_{213}^{-1} \cdot \tilde{\mathcal{R}}_{12}(\lambda + h^{(3)}),$$

where $\tilde{\Phi}$ is the co-associator of propositions 2, 2′ and $\tilde{\mathcal{R}}(\lambda)$ denotes $\mathcal{R}'(\lambda)$, $\mathcal{R}_0(\lambda)$, respectively. Moreover, applying $(T \otimes 1)((1 \otimes T)(T \otimes 1)$ to equation (8.7) it is easily seen that $\mathcal{R}^T(\lambda)$ also satisfies the above opposite quasi-dynamical QYBE but with respect to the opposite co-associator Φ^T of proposition 1.

We anticipate that the quasi-dynamical QYBE will play an important role in obtaining elliptic solutions to the QQYBE from trigonometric ones via twisted QUEs. Of particular interest is the quasi-dynamical QYBE for elliptic quantum groups.

References

[1] Altschuler D and Coste A 1992 Quasi-quantum groups, knots, three manifolds and topological field theory *Commun. Math. Phys.* **150** 83–107 (*Preprint* hep-th/9202047)

[2] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[3] Dijkgraaf R, Pasquier V and Roche P 1990 Quasi-quantum groups related to orbifold models *Proc. Int. Coll. on Modern Quantum Field Theory* (Bombay: Tata Institute)

[4] Dijkgraaf R, Vafa C, Verlinde E and Verlinde H 1989 The operator algebra of orbifold models *Commun. Math. Phys.* **123** 485–526

[5] Drinfeld V G 1990 On almost cocommutative Hopf algebras *Leningrad Math. J.* **1** 321–42

[6] Drinfeld V G 1990 Quasi-Hopf algebras *Leningrad Math. J.* **1** 1419–57

[7] Drinfeld V G 1991 On quasitriangular quasi-Hopf algebras and a group closely connected with $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ *Leningrad Math. J.* **2** 829–60

[8] Enriquez B and Felder G 1998 Elliptic quantum group $E_{\tau,\eta(\mathfrak{s}l_2)}$ and quasi-Hopf algebras *Commun. Math. Phys.* **195** 651–89 (*Preprint* q-alg/9703018)

[9] Foda O, Iohara K, Jimbo M, Kedem R, Miwa T and Yan H 1994 An elliptic quantum algebra for $\widehat{\mathfrak{s}l_2}$ *Lett. Math. Phys.* **32** 259–68 (*Preprint* hep-th/9403094)

[10] Felder G 1995 Elliptic quantum groups *Proc. XIth ICMP (Paris, 1994)* pp 211–8

[11] Fronsdal C 1997 Quasi-Hopf deformations of quantum groups *Lett. Math. Phys.* **40** 117–34 (*Preprint* q-alg/9611028)

[12] Gould M D, Zhang Y-Z and Isaac P S 2000 Casimir invariants from quasi-Hopf (super) algebras *J. Math. Phys.* **41** 547–68 (*Preprint* math.QA/9811052)

[13] Gould M D, Zhang Y-Z and Isaac P S 2001 On quasi-Hopf superalgebras *Commun. Math. Phys.* **224** 341–72 (*Preprint* math.QA/9811062)

[14] Jimbo M, Odake S, Konno H and Shiraishi J 1999 Quasi-Hopf twistors for elliptic quantum groups *Transform. Groups* **4** 303–27 (*Preprint* q-alg/9712029)

[15] Links J R, Gould M D and Zhang Y-Z 2000 Twisting invariance of link polynomials derived from ribbon quasi-Hopf algebras *J. Math. Phys.* **41** 5020–32 (*Preprint* math.QA/9904069)

[16] Mack G and Schomerus V 1991 Quasi quantum group symmetry and local braid relations in the conformal Ising model *Phys. Lett.* B **267** 207–13

[17] Majid S 2000 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)

[18] Reshetikhin N 1990 Quasitriangle Hopf algebras and invariants of links *Leningrad Math. J.* **1** 491–513

[19] Sweedler M E 1969 *Hopf Algebras* (New York: Benjamin)

[20] Zhang Y-Z and Gould M D 1999 Quasi-Hopf superalgebras and elliptic quantum supergroups *J. Math. Phys.* **40** 5264–82 (*Preprint* math.QA/9809156)